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# Cross-layers cascade in multiplex networks

Zhaofeng Li<sup>1,2</sup> · Fuhan Yan<sup>1,2</sup> · Yichuan Jiang<sup>1,2</sup>

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**Abstract** The study of information cascade in multiplex networks, where agents are connected by using multiple linking types, has received increasing attention. Compared with the cascade in simplex networks, a noticeable characteristic of the cascade in multiplex networks is that information may be spread between multiple layers. In this study, we focus on the cross-layers cascade, which helps clarify two opposing opinions about the information cascade in multiplex networks: multiplexity can speed up or slow down information cascade. The linear threshold model is generalized into multiplex networks as conjoint agents become active, if the influences of active neighbors in any layer reach a predefined threshold. The preconditions and reasons for the slow-down and speed-up phenomena are discussed using four representative case studies and theoretical analyses. Next, analytical results are validated by using extensive simulations in which the multiplex networks are generated by random, small-world and scale-free network models. It is found that the slow-down phenomenon emerges due to the obstruction of cross-layers cascade which connects the distributed shortest path in multiple layers and the inhibitory effect of negative influence. Conversely, extra short paths or rapid spreading in one additional layer can facilitate the cascade process in existing networks, respectively. Extensive simulations also show that multiplex networks consisting of different network models are more competent for the cascade process compared with multiplex networks generated by a single network model. In conclusion, the concept of cross-layers cascade may elucidate the additional study of information spreading in multiplex networks.

**Keywords** Information cascade · Multiplex networks · Cross-layers · Linear threshold model · Multi-agent systems

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# 1 Introduction

Cascade is an interesting phenomenon referring to a global diffusive process of a local effect, which is initialized by one node or a small fraction of nodes in a network [1,2]. In social life, the adoption of innovation and social norms [3,4], propagation of behaviors and strategies [5–7], and catastrophic spreading of failures and epidemics [8–11] are well-known cascade phenomena. These diffusive processes are often studied by using the linear threshold model [2,12,13] in which a node becomes active if the influences of active neighbors exceed a predefined value. Traditional studies generally analyzed the cascade process in simplex networks where linking types between nodes are identical. Recently, more studies have realized that real social networks contain multiple-layer structures mainly because social agents are connected by multiple linking types [14–18]. In multiplex networks, an agent can transfer information between layers in addition to spreading information within each layer. For instance, a person can share topics in real life communication to online social networks or post his/her tweets (from Twitter) to Facebook. To the best of our knowledge, few studies have formally described the details of cross-layers cascade, which helps to analyze the complicated effects of multiplexity on cascade processes.

Based on some solid theoretical foundations, many studies have analyzed the cascade process in multiplex networks and compared it with the cascade process in a simplex network. One general opinion is that multiplexity can speed up the cascade process [18–21]. References [18] and [19] reveal the dramatic effect of conjoining two entirely different networks based on the velocity and size of the information cascade. In [20], a superdiffusive behavior is discussed, which means that the cascade process in multiplex networks is faster than the cascade process in any disjointed layers. Meanwhile, Brummitt et al. show that multiple networks are more vulnerable to global cascades than simplex networks and suggest that adding or removing sparse layers in existing multiple layers is a feasible way to control the cascade process in multiplex networks [21]. In a word, it is generally accepted that multiplexity provides more feasible paths for information cascade. Indeed, people receive vast amounts of information quickly from multiple channels every day, and many new fashions in online social networks have become hot topics in real life.

However, according to some real data, cascade processes always turn out slow as information spreads on a topologically inefficient path, which means the propagation path is considerably longer than the shortest link between two randomly selected nodes in large scale networks [22–24]. It is known that the speeds of spreading information on distinct linking types are different [25,26]. In real networks, information flows easily on part of the edges, while the cascade tends to be dampened in the remaining part. For example, Tang et al. first identify the relationships among social network users and then utilize the different efficiencies of the relationships on the cascade processes to maximize the final adoption rates of the different products [26]. Therefore, multiplexity may cause a slow-down phenomenon because information selectively propagates on networks and cannot be freely transferred from one layer to conjoining layers.

In this paper, we focus on the cross-layers cascade to understand and explain how the cascade process in multiplex networks is slowed down or sped up. Two features of cross-layers cascade are proposed: the mapping relationship conjoins multiple layers and provides the transfer paths; and the vertical transfer coefficient quantifies the influences of nodes varied in multiple layers. The two features help clarify the details of the information cascade in multiplex networks. Then, the linear threshold model is generalized to multiplex networks in which a node becomes active if the influences of active neighbors in any layer reach a

predefined threshold [21]. Based on the cross-layers cascade, different cascade processes are scrutinized in four representative case studies, and the preconditions for slow-down and speed-up phenomena are discussed. With the aid of theoretical analyses, we also show that the final size and velocity of the cascade process in the underlying simplex networks are larger than the ones in multiplex networks under certain conditions. Moreover, extensive simulations are utilized to validate the analytical results above and present other features of the cascade process in multiplex networks. In this paper, the topologically inefficient paths of information cascade in social networks [22] are analyzed based on the perspective of the multiplex networks. The obstruction of cross-layers cascade which connects the distributed shortest path in multiple layers and the inhibitory effect of negative influence are the underlying reasons. We hope that the introduction of cross-layers cascade can provide new insights into the study of information cascade in multiplex networks.

The rest of the paper is organized as follows. In the following section, an overview of the related literature is given. We outline the details of cascade across layers and the cascade model in Sect. 3. In Sect. 4, we analyze the slow-down and speed-up phenomena in multiplex networks. Simulation results and analyses are presented in Sect. 5. In Sect. 6, we present our conclusions and note the future outlook of our research.

## 2 Background and related work

In this section, we briefly compare the theoretical analyses and empirical studies of the cascade process and highlight the advantages of the agent-based modeling and simulation method [16, 27] used in this paper.

### 2.1 Theoretical analyses

The cascade process is the cumulative result of each agent's behavior based on the interaction environment. The interactive behaviors between social agents or people's decisions in a cascade process are generally formulated by different cascade models. The linear threshold model [2] is a widely analyzed model in addition to the independent cascade model [13] and epidemic models [28, 29] in which an agent decides to disseminate the information according to a given probability. These models are very concise and convenient for theoretical analyses in the framework of different types of social networks. Reference [18] finds that global cascade can emerge in multiplex networks even if no cascade processes take place in each individual layer with the epidemic model. By using mean-field approaches, the final size and critical condition of the global cascade are both quantified. The construction of a supra-Laplacian matrix is proposed in [20] to represent the entire structure of multiplex networks. After the analyses of eigenvectors and eigenvalues of the supra-Laplacian matrix, a superdiffusive behavior is discussed, which means that the cascade process in multiplex networks is faster than the cascade process in any disjointed layers. In [19], Buldyrev finds that the multiplex networks consisting of different infrastructure systems are more vulnerable. From the perspective of phase transition in a complex system, Brummitt and his coworkers report that the cascade process can be facilitated by both splitting a simplex network into multiplex networks and by combining different layers [21]. The main objects of this theoretical study are to establish the formula for estimating the size of the cascade process and give the critical condition under which the cascade process dies out quickly. A more comprehensive review can be found in [30, 31].

The results of using a theoretical perspective are provable and rigorous under given conditions and cascade models. However, there are still two main problems. First, the cascade velocity, namely, the time to reach a stationary state (global cascade), is largely overlooked. This parameter is of profound practical significance. For instance, in virtual marketing, the influence of an innovation product should be spread to the maximized scope in the shortest time [32]. As Salehi et al. [30] suggest, the issue of cascade velocity needs more attention. Second, most theoretical studies focus on the differences of the cascade processes in simplex and multiplex networks and ignore the differences between diverse multiplex networks. For example, in [18] and [21], all layers of multiplex networks are Erdős–Rényi networks. In [33] and [34], the Erdős–Rényi network and scale-free network are conjoint. In fact, different network models have diverse characteristics, such as a smaller mean-shortest path of the Erdős–Rényi model [35], higher clustering coefficient of the small-world model [36] and robustness with random failures of the scale-free model [37]. Using agent-based simulations, one of our goals is to analyze the effects of the multiplex structure consisting of different networks models.

## 2.2 Empirical studies

An empirical study can clarify the cascade process more appropriately and requires a specific example of the cascade process, massive quantities of data and efficient tools for data analysis. Moreover, many empirical conclusions validate or query the hypotheses proposed by theoretical analyses. An excellent example is the online experiment performed by Centola, who validates the hypothesis that a random network is less effective for behavioral diffusion than a small-world network, when behaviors require social reinforcement [5]. The slow-down phenomenon of information cascade occurs in the framework of a simplex mobile phone network [22,23]. The bursty activity patterns of individuals and the dynamical weights of links are proposed to be the main reasons for the slowing down phenomenon [22,23]. In fact, each individual has diverse types of relationships with neighbors in the mobile phone network. The empirical data for mobile phone communication is not analyzed in the form of multiplex networks because the relationships between people are more difficult to identify compared with the frequency and duration of the phone call.

Currently, real field data concerning information cascade in multiplex social networks is still very rare. How to identify the cross-layers cascade is the main difficulty because online social networks belong to different companies. Very recently, some empirical studies have tried to discover the cross-layers relationship of account names in different online social networks. With the aid of machine learning technology, the spelling of account names [38,39], overlap of linking friends [40] and behavioral features [41] in different social networks are essential factors for identifying the cross-layers relationship. Other researchers try to construct multiplex networks based on the reported quality of edges. According to the movie genres, Nicosia and Latora [42] successfully generate a 28-layer multiplex network based on the Internet Movie Database (IMDb). Meanwhile, the relationships between Indonesian terrorists are the main standards of the multiplex data set [43]. Because of the noise and time sampling in real data, an empirical study is generally costly (huge quantities of data and excellent processing ability) and may not directly analyze the effect of a particular network structure on information cascade.

## 2.3 Agent-based models

Agent-based modeling and simulation is an economical and powerful approach to investigate the cascade process because it offers a feasible way to generate complex structures of social networks and reproduce human behavior patterns [16, 27, 44–47]. For instance, Anshelevich and his coworker use agent-based models to determine the optimal seed sets of inhabitants, which can lead to the best evacuation procedure during emergency warnings [45]. Moreover, agents can be assumed to have the ability of reinforcement learning and decision-making [46, 47] in multi-agent systems. Exhaustive advantages of agent-based models have been discussed in [16, 27] and [44] from different perspectives. In this paper, we use the agent-based modeling method to give a simple definition of cross-layers cascade and to easily construct diverse types of multiplex networks. Then, analytical results and different phenomena of cascade processes are validated by using extensive agent-based simulations.

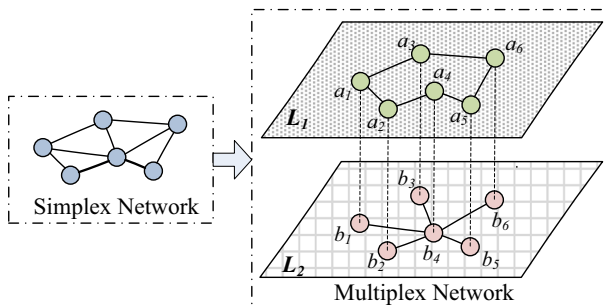
## 3 Model outline

In this section, we first give the model of multiplex social networks and describe the features of the cross-layers cascade. Then, we generalize the linear threshold model to multiplex networks.

### 3.1 Multiplex networks

A network is always formulated as a graph  $G = (V, E)$  in which  $V$  is the set of nodes and  $E$  is the set of edges linking nodes [26]. Agent and node are interchangeable concepts as follows. In this paper, we use  $G$  to represent the structure of an underlying simplex network of multiplex networks. Multiple and parallel graphs are usually used to represent multiplex networks [20, 21]. According to the categories of linking types  $\{l_1, l_2, \dots, l_n\}$  [26], multiplex networks which contain  $n$  layers are denoted by  $\{L_1, L_2, \dots, L_n\}$ , as shown in Fig. 1.

For simplicity,  $L_1, L_2, \dots, L_n$  also represent the set of agents in each layer. The letter  $i$  is the identification of agent, and agents with the same identification are conjoint in multiple layers. Meanwhile, the factor  $v_i^{L_n}$  indicates the agent in  $L_n$  and  $v_i^l$  represent the set of conjoint agents  $\{v_i^{L_1}, v_i^{L_2}, \dots, v_i^{L_n}\}$  in multiple layers. For the sake of simplicity and clear description, we use  $a_i$  and  $b_i$  to denote the two conjoint agents in  $L_1$  and  $L_2$ . Agent has binary states:  $a_i^1$  ( $a_i^s = 1$ ) means  $a_i$  is active, and  $a_i^0$  ( $a_i^s = 0$ ) means current state is inactive.  $\Omega_{a_i}^{L_1}$  is the set of nodes linking to  $a_i$  in  $L_1$  ( $a_i \in L_1, \Omega_{a_i}^{L_1} \subseteq L_1$ ). The degree of  $a_i$  in  $L_1$  equals  $|\Omega_{a_i}^{L_1}|$ .



**Fig. 1** Illustration of simplex and multiplex networks

It is worth noting that there are two methods to generate multiplex networks: one is “splitting” [15,26]; and the other is “combining” [18,19,21]. The splitting method means that nodes and edges of  $G$  are distributed in  $L_1, L_2, \dots, L_n$  as realizing the diverse linking types of a simplex network. The combining method indicates that  $L_1, L_2, \dots, L_n$  are conjoint by the relationships between nodes in different layers if one agent takes part in interdependent cascade processes in different networks. No matter how multiplex networks are generated, the degree of agent in each layer is no larger than the one in  $G$ .

## 3.2 Cross-layers cascade

### 3.2.1 Mapping relationship

The mapping relationship indicates the dependence of states between agents. We first define the mapping relationship of agents in a single layer. The symbol “ $\rightarrow$ ” is used to represent the correlation between agents.  $\Omega_{a_i}^{L_1} \rightarrow a_i$  means the state of  $a_i$  depends on the neighbors of  $a_i$ . If  $a_i \rightarrow a_j$  and  $a_j \rightarrow a_i$ , then  $a_i \leftrightarrow a_j$ . In a single layer, the mapping relationship is like the directed edge in graph theory [48]. Then, the mapping relationships of agents between layers are given.

**Definition 1**  $b_i$  is injective to  $a_i$ . If  $a_i \in L_1, \exists b_i \in L_2$  and  $b_i$  is unique, such that  $b_i \rightarrow a_i$  and  $b_i$  is not corresponding to  $a_i$

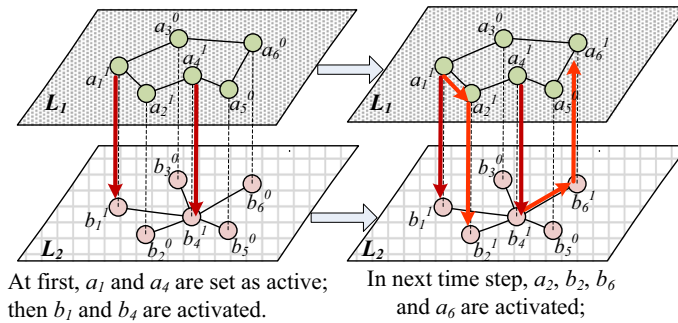
**Definition 2** If  $a_i \in L_1, \exists b_i \in L_2$  and  $b_i$  is unique, and if  $b_j \in L_2, \exists a_j \in L_1$  and  $a_j$  is unique, such that  $a_i \leftrightarrow b_i$  and  $b_j \leftrightarrow a_j$ , then the mapping relationship between  $L_1$  and  $L_2$  is bijective.

**Definition 3** The mapping relationship between  $L_1$  and  $L_2$  is multi-bijective. If  $\forall a_i \in L_1, \forall b_j \in L_2$ , and  $\exists b_i \in L_2, \exists a_j \in L_1$ , such that  $\{b_i \dots\} \rightarrow a_i, \{a_j \dots\} \rightarrow b_j$ .

The injective relation is unidirectional and provides a foundation for other mapping relationships. The multi-bijection relation is the main characteristic of interdependent infrastructure systems. For example, one power station can supply several nodes in the Internet communication network, and several power stations may communicate through one or more communication nodes [8, 19]. Many online social networks, email networks and mobile communication networks can be considered as bijective multiplex social networks. In this paper, we focus on the cascade process under the bijective relationship between layers. The symbol  $\Phi$  is introduced:  $\Phi_{a_i}^{L_2}$  is the set of agents in  $L_2$  that are  $\Phi_{a_i}^{L_2} \subseteq L_2$  and  $\Phi_{a_i}^{L_2} \rightarrow a_i$ . For example,  $\Phi_{a_1}^{L_2} = \{b_1\}$  in Fig. 1.

The mapping relationship conjoins multiple layers and provides the paths for cross-layers cascades. Figure 2 shows that the cascade process in multiplex networks consists of a cascade across layers and a cascade on each layer. At first,  $a_1$  and  $a_4$  are set as active. Then, cross-layers cascades take place:  $b_1$  and  $b_4$  are activated. In next time step,  $a_2$  is activated by  $a_1$  in  $L_1$ , and  $b_6$  becomes active due to  $b_4$  in  $L_2$ . At last,  $b_2$  and  $a_6$  are activated, even though  $b_2$  is not linked to  $b_1$  and  $a_6$  is isolated from  $a_4$ . Therefore, without the consideration of cross-layers cascade, a cascade on each layer is difficult to analyze.

In real multiplex social networks, it may take different time intervals to transfer information between multiple layers. Time intervals of cross-layers cascades lead to asynchronous cascade processes [49] in multiple layers: when  $b_i$  is activated by its neighbors in  $L_2$ , its mapping agent  $a_i$  in  $L_1$  has been in an active state for several time steps. In this paper, the time interval of the cross-layers cascade is assumed to be zero for the sake of simplicity. Only a simple



**Fig. 2** Illustration of cascade process in multiplex networks

case study is given to describe how different time intervals of cross-layers cascades further slow cascade processes in multiplex networks.

### 3.2.2 Vertical transfer coefficient

The vertical transfer coefficient proposed in this paper denotes the diverse influences of agent  $v$  in multiple layers. The symbol  $\lambda$  represents the vertical transfer coefficient, and  $\lambda$  can be positive or negative.  $\lambda_{b_i}^{a_i}$  means the influence of  $b_i$  in  $L_2$  transferred to  $a_i$  in  $L_1$  if  $a_i \in L_1$ ,  $b_i \in L_2$  and  $a_i^s \leftrightarrow b_i^s$ . Then, the influence of one agent in a certain layer is the sum of the vertical transfer coefficients of conjoint agents mapped from other layers. For example, in Fig. 2, the influence of  $b_4$  received by  $b_6$  is calculated by  $\lambda^{b_4} \times b_4^s$  and equals  $\lambda_{a_4}^{b_4} \times b_4^s$ . For simplicity, if  $\forall a_i \in L_1$  and  $\Phi_{a_i}^{L_2} = \{b_i\}$ ,  $\lambda_{b_i}^{a_i}$  are identical, then  $\lambda_{L_2}^{L_1}$  equals  $\lambda_{b_i}^{a_i}$  and represents the strength of  $L_2$  mapping to  $L_1$ . Then,  $\lambda^{L_i}$  is named as the vertical transfer coefficient of  $L_i$ :

$$\lambda^{L_i} = \sum_{i \neq j}^n \lambda_{L_j}^{L_i}$$

In this case, the influence of an agent in a certain layer equals the vertical transfer coefficient of this layer. Vertical transfer coefficient means the weight of the mapping relationship between layers. Some related studies have defined similar parameters in multiple networks. In [50], the content-dependent parameters indicate different weighted links in each layer. Generally, an agent shows positive influences on the activations of neighbors if  $\lambda > 0$  and inhibits the cascade process if  $\lambda < 0$ .

### 3.3 Cascade model

In this section, we generalize the linear threshold model to multiplex networks in which a node activates if the influences of neighbors in any layer are larger than the threshold [21]. In the classical linear threshold model, node  $v$  randomly chooses a threshold  $\theta_v$  from the interval  $[0, 1]$ . Node  $v$  is linked with positive weight edges of which the sum is less than one. Node  $v$  becomes active if the sum of the weight edges linking to an active neighbor exceeds the threshold  $\theta_v$  [12, 13]. Watt's threshold model [2] expands the linear threshold model, and node  $v$  is activated if the fraction of active neighbors is larger than  $\theta_v$ , ignoring the weights of edges.

The principle of the linear threshold model is that the activation of a node or the decision of people to diffuse certain information needs reinforcements from multiple neighbors [5]. In real life or in online social networks, it may be impossible for a person to estimate the number of all neighbors and calculate the fraction of active linking agents because of different social ties, familiarities, or communication intervals. It is more feasible to estimate the sum of influences of interactive neighbors. Therefore, node  $v$  in a simplex network becomes active if

$$\tau_v \geq \theta_v.$$

where  $\tau_v$  is the sum of influences of neighbors of node  $v$ . The magnitude of an agent's influence in a simplex network is one unit. The factor  $\tau_v^{L_i}$  indicates the sum of influences of active neighbors of node  $v$  in layer  $L_i$ . Then, in multiplex networks, node  $v$  becomes active if

$$\begin{aligned} \max_{i=1,\dots,n} (\tau_v^{L_i}) &\geq \theta_v. \\ \tau_v^{L_i} &= \lambda^{L_i} \left( \sum \Omega_v^{L_i} \right). \end{aligned}$$

The item  $\sum \Omega_v^{L_i}$  indicates the number of active neighbors of node  $v$  in  $L_i$ . The factor  $\tau_v^{L_i}$  is simplistically calculated because of the assumption of identical vertical transfer coefficients of agents in each layer. If identical vertical transfer coefficients of agents are different,  $\tau_v^{L_i}$  should be obtained according to the definition. For example, agent  $a_i$  will be activated in  $L_1$  if  $\tau_{a_i}^{L_1} = \sum_{j \in \Omega_{a_i}^{L_1}} \lambda^{a_j} \cdot a_j^s \geq \theta_{a_i}$ . With our generalized cascade model, an agent can easily receive enough influences and widely spread its influence if the degree of agent is large. This assumption accords with reality. For example, hub nodes which contact a large number of nodes in social networks are more susceptible to the viruses and can easily infect a significant fraction of nodes in networks [10].

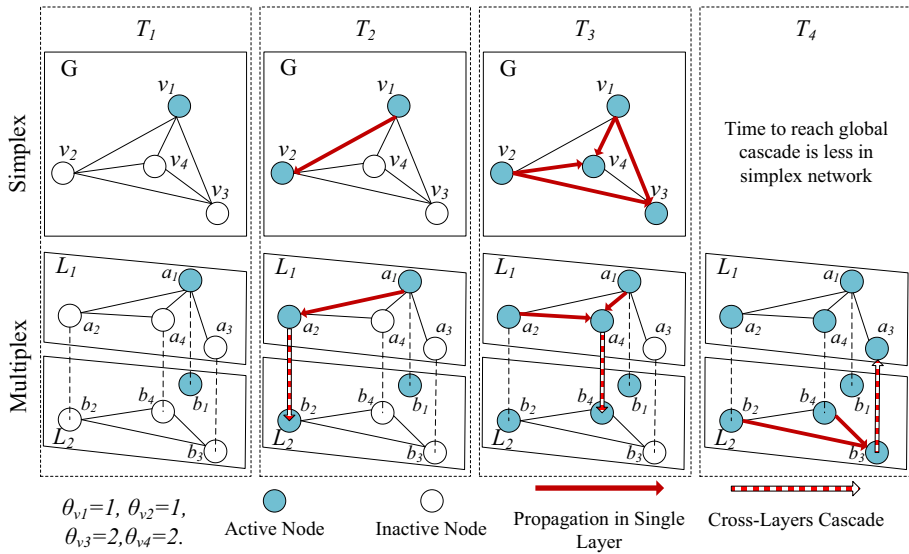
The cascade process heavily depends on the thresholds of nodes. Many studies have found the critical thresholds of global cascades in several types of networks [18, 20, 21], but reliable formulas of threshold distributions according to different types of collective human behaviors still remain unknown [5]. In this paper, thresholds are assumed to follow normal and uniform distributions, while the cascade condition and critical threshold are not the main aims.

## 4 Analysis of cascade process

In this section, cascade processes in multiplex networks are briefly analyzed in four case studies with the aid of cross-layers cascade. The reasons and preconditions for slow-down and speed-up phenomena are discussed. Then, the differences between the cascade processes in the simplex network and multiplex networks are theoretically depicted.

### 4.1 Case study

Simplex network  $G$  is supposed to be a complete graph on four vertices, which are denoted by  $v_1, v_2, v_3$  and  $v_4$ . Thresholds of  $v_1$  and  $v_2$  are one, while  $v_3$  and  $v_4$  become active if at least two neighbors are active. Node  $v_1$  is initialized as an active state. It takes three time steps to reach a global cascade in  $G$ , and the propagation paths are all shortest, as shown in Fig. 3.



**Fig. 3** Illustration of slow-down phenomenon in multiplex networks

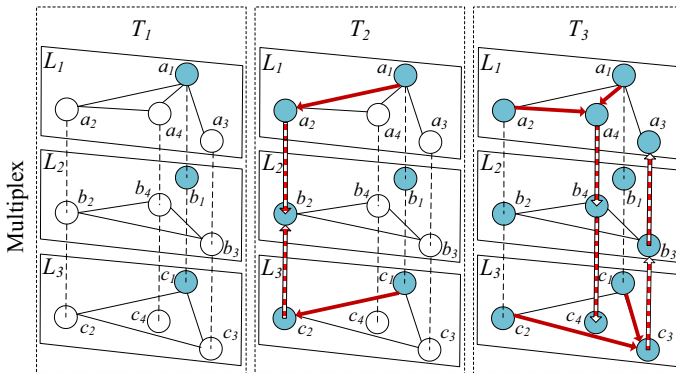
#### 4.1.1 Case one

In the first case study, a simplex network is split into two-layer multiplex networks [15,26], and the slow-down phenomenon in multiplex networks emerges.  $\lambda_{L_2}^{L_1}$  and  $\lambda_{L_1}^{L_2}$  are both one unit.

As shown in Fig. 3, each layer contains part of edges in  $G$ , and some agents cannot be activated in a certain layer because of isolation or an insufficient number of neighbors. For example, the activations of  $b_2$  and  $b_4$  depend on cross-layers cascades from  $a_2$  and  $a_4$ , which can both become active following the shortest paths in  $G$ . Meanwhile,  $a_3$  cannot be activated by neighbors in  $L_1$ , which only contains the first part of the shortest path to active  $v_3$  in  $G$ . The cross-layers cascade from  $a_3$  to  $b_3$  is obstructed because  $a_3$  is surrounded by insufficient active nodes in  $L_1$ . One additional step is needed to activate  $b_3(a_3, v_3)$  after the activation of  $b_4$ . However,  $v_3$  and  $v_4$  can be activated by  $v_1$  and  $v_2$  simultaneously in  $G$ .

Therefore, the shortest path in a simplex network is distributed in multiple layers, and the obstruction of cross-layers cascade is the main reason for the slow-down phenomenon. If an intermediate agent lacks sufficient neighbors in one layer, which contains the first part of the shortest path, the cross-layers cascade from the intermediate agent cannot take place, and information spreading on the shortest path is blocked. Then, downstream nodes will be activated by information spreading on other longer paths, which are topologically inefficient. As a result, the slow-down phenomenon of the cascade process in multiplex networks emerges compared with the information spreading in a simplex network. With the consideration of the time interval of the cross-layers cascade, the global cascade will be further postponed. For example, if the cascade from  $a_2$  to  $b_2$  falls behind the cascade from  $a_4$  to  $b_4$ , the activation of  $b_3$  will be delayed.

It needs to be mentioned that traditional studies on social networks focused on the topology of agents' interactions without the consideration of the types of interactions. In other words, the topologically inefficient path found in empirical data may be the most efficient path in



**Fig. 4** Illustration of facilitation of additional layer in multiplex networks

the framework of multiplex networks. For instance, in the mobile communication network [22,23], mobile phone users constitute the underlying simplex network, which can be split into multiplex networks according to different linking types (multi-relation) [15,22]. Therefore, whom a person calls must be related to the topic. Then, the edges of the shortest path (whom a person knows) between two randomly selected nodes are distributed in multiple layers, and many paths of cross-layers cascades are added. In social life, people consciously block cross-layers cascades because people do not communicate with every acquaintance about new fashions or share all new messages in online social networks.

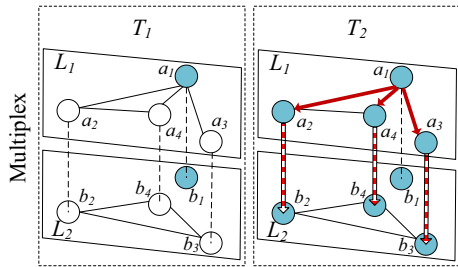
#### 4.1.2 Case two

The second case study describes the facilitation of an additional layer in multiplex networks, as shown in Fig. 4.  $\lambda^{L_1}$ ,  $\lambda^{L_2}$  and  $\lambda^{L_3}$  are all one unit. Compared with the multiplex networks in Fig. 3, the structure of the underlying simplex network remains the same, but  $L_3$  provides an extra path to activate  $c_3$ , which is shorter than the existing path in  $L_2$  and improves the speed of the information cascade. By cross-layers cascade, mapping nodes in  $L_1$  and  $L_2$  are successively activated. However, the time to reach global cascade is still  $T_3$ .

It is worth noting that the acceleration of the cascade process in multiplex networks emerges compared with the case in which the network contains disjoint multiple layers [18,20]. In Fig. 4 for instance, a global cascade will not appear in any layer if the three layers are disjoint. By conjoining different networks, the increase of nodes' degrees and the structural changes in an underlying simplex network are the probable reasons for the speed-up phenomenon. Meanwhile, references [18] and [20] made no comparison between conjoint multiple layers and the underlying simplex network.

In [21], Watt's threshold model is generalized to multiplex networks, and a node is activated if the proportion of active neighbors exceeds the threshold in any layer. The facilitation of multiplexity depends on the property of Watt's threshold model. A single layer may be unsusceptible to global cascade because of the constraint of network connectivity: a sparse network lacks global connectivity; and a node is always surrounded by an insufficient proportion of active neighbors in a dense layer. By coupling together or splitting a sparse layer from a dense network, most nodes easily become active in the sparse layer, and the influences of active nodes are widely spread because of the high connectivity in the dense layer.

**Fig. 5** Illustration of speed-up phenomenon in multiplex networks



#### 4.1.3 Case three

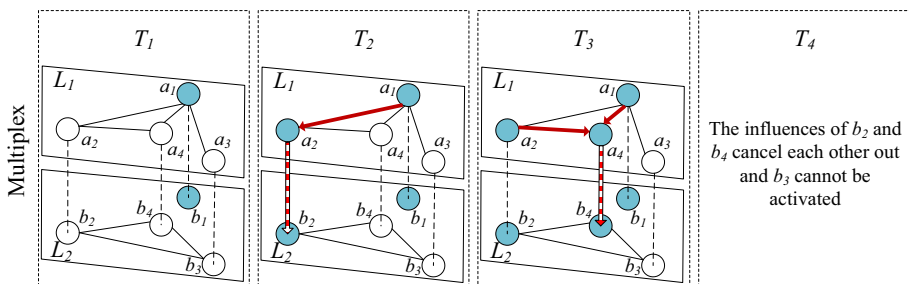
The third case study shows the speed-up phenomenon in multiplex networks caused by a rapid cascade in one layer (large transfer coefficient).  $\lambda_{L_1}^{L_1}$  is set as two, and  $\lambda_{L_1}^{L_2}$  is still one. In this case, the influence of  $a_1$  equals two active agents. Then, agents in  $L_1$  are activated by  $a_1$  at  $T_2$ , and agents in  $L_2$  become active because of cross-layers cascades simultaneously. The large value of transfer coefficient induces a rapid cascade process in one layer and has a positive effect on global cascade.

The third case study is simple but shows an interesting and common phenomenon in daily life. It is supposed that  $L_1$  denotes Facebook or Twitter,  $L_2$  is the word of mouth communication network for acquaintances [18], and  $a_1$  represents a famous music star. Thus, Fig. 5 means that  $a_2$ ,  $a_3$  and  $a_4$  are “fans” of  $a_1$  but have no personal relationship with  $a_1$  (the music star is isolated in  $L_2$ ). If  $a_1$  uploads a new song on an online social network,  $a_2$ ,  $a_3$  and  $a_4$  know it immediately without talking to each other in  $L_2$ . Indeed, new fashion spreads very fast and can become a hot topic mostly because of the rapid cascade process in online social networks instead of the one in word of mouth communication.

On the other hand, if the vertical transfer coefficient of a certain layer is considerably larger, the cascade processes in other layers may be inhibited. In Fig. 5, no nodes are activated in  $L_2$  because the active states are all vertically transferred from  $L_1$ . However, if  $a_3$  is not a fan of  $a_1$  in  $L_1$ ,  $b_3$  will know about the new song from the conversations between  $b_2$  and  $b_4$ .

#### 4.1.4 Case four

In the fourth case study, only  $\lambda_{a_4}^{b_4}$  is set as  $-1$ , and the other parameters are the same as the ones in the first case study. In Fig. 6, the slow-down phenomenon caused by the negative vertical transfer coefficient is depicted. It can be found that the cascade process is the same



**Fig. 6** Illustration of inhibitory effect of negative vertical transfer coefficient

as the one in Fig. 3 before  $T_4$ . Agent  $a_4$  is activated in  $L_1$  at  $T_3$ , and  $b_4$  is activated by the cross-layer cascade. Because of the negative influence of active  $b_4$  on inactive  $b_3$ , the positive influence of  $b_2$  is canceled out, and the sum of influence on  $b_3$  is zero. Different from the foe or negative relation in the studies of social balance [51, 52] and a signed network [53], the negative value of the vertical transfer coefficient indicates the inhibitory effect of an agent on the activation of linking neighbors because more positive influences are needed to cancel out the negative influences.

The inhibitory effect of the negative vertical transfer coefficient can be clearly depicted in the case of the adoption of an innovation product [13, 54]. The adoption of an innovation is a composite result of complex spreading phenomena in many conjoint social networks, such as the advertisement or sales promotion in online social networks [55] and talks in word-of-mouth communication networks between friends or acquaintances [13]. If a person is not satisfied with the innovation, which is bought because of the promotion in online social networks, he/she may spread a negative appraisal in many networks. In this case, this negative opinion can interfere with the adoptions of the innovation of other people and inhibit the cascade process.

## 4.2 Mathematical description

It is useful to theoretically understand the differences between the cascade processes in the simplex network and multiplex networks. The final size and velocity are two important features of the cascade process. At first, we prove that the final size of the cascade process in multiplex networks is no larger than the one in a simplex network with the same parameter values of agents' influences and threshold distributions. Then, the times to reach global cascade in multiplex networks is shown to be not shorter than the one in a simplex network.

**Lemma 1** *If the cascade process in a simplex network  $G = (V, E)$  reaches the stationary state, the final state is also stationary in any multiplex network  $\{L_1, L_2, \dots, L_n\}$  split from the  $G = (V, E)$  with the one unit influence of each agent.*

*Proof* The Lemma follows because the number of agents in each layer equals  $|V|$ , and each layer contains part of or all of the edges in  $E$ . The stationary state means that no more agents can be activated. When the cascade process in  $G$  is terminated, the following condition is satisfied for every inactive node:

$$\tau_v < \theta_v.$$

Then, if the states of agents in multiplex networks are assigned the same states of nodes with the identical identifications in a simplex network, the following condition is also satisfied for all conjoint agents that are not activated:

$$\tau_v^{L_1} < \theta_v; \tau_v^{L_2} < \theta_v \cdots \tau_v^{L_n} < \theta_v.$$

This indicates that no more agents can be activated, and the final state in a simplex network is also stationary in multiplex networks no matter how the edges are distributed.  $\square$

**Lemma 2** *If the cascade process in multiplex networks  $\{L_1, L_2\}$  reaches the stationary state, the final state may not be stationary in the underlying simplex network  $G = (V, E)$  with the one unit influence of each agent.*

*Proof* When the cascade process in  $\{L_1, L_2\}$  is terminated, the influences received by the inactive conjoint agents satisfy

$$\begin{aligned}\tau_v^{L_1} &< \theta_v; \\ \tau_v^{L_2} &< \theta_v.\end{aligned}$$

Then, for the nodes in the underlying simplex network, the following condition is satisfied:

$$\tau_v \leq \tau_v^{L_1} + \tau_v^{L_2}.$$

By removing the redundancies of active conjoint agents,  $\tau_v = \tau_v^{L_1} + \tau_v^{L_2} - \alpha$ . Supposing that the distributions of  $\tau_v^{L_1}$  and  $\tau_v^{L_2} - \alpha$  are  $f_1(\tau_v^{L_1})$  and  $f_2(\tau_v^{L_2} - \alpha)$  within the interval  $[0, \theta_{max}]$  where  $\theta_{max}$  is the maximum threshold of inactive agents, the function  $f(\tau_v)$  is the distribution of  $\tau_v$  and is derived by

$$f(\tau_v) = \int_0^{\theta_{max}} f_1(\tau_v^{L_1}) f_2(\tau_v - \tau_v^{L_1}) d\tau_v^{L_1}.$$

Then, when  $\tau_v$  is set as the maximum threshold  $\theta_{max}$ ,

$$f(\theta_{max}) = \int_0^{\theta_{max}} f_1(\tau_v^{L_1}) f_2(\theta_{max} - \tau_v^{L_1}) d\tau_v^{L_1} \geq 0.$$

The probability density distribution when  $\tau_v = \theta_{max}$  does not identically equal zero. It indicates that the influences received by the nodes in the underlying simplex network may satisfy  $\tau_v \geq \theta_{max}$ . Therefore, the final state may not be stationary in the underlying simplex network because some additional nodes can be activated by combining the multiplex networks into the underlying simplex network.  $\square$

For the sake of simplicity, Lemma 2 is based on two-layer multiplex networks because the distribution function of an n-dimensional random vector is very complex. However, the final state in n-layer multiplex networks may also not be stationary in the underlying simplex network. Based on Lemmas 1 and 2, we can derive Corollary 3.

**Corollary 3** *The final size of the cascade process in any multiplex network is no larger than the one in the underlying simplex network with the one unit influence of each agent.*

The time to reach the stationary state is used to represent the cascade speed in networks. The size of the stationary state in Lemma 4 is restricted to the global cascade because the time to reach the stationary state cannot represent the velocity of the cascade process if the cascade process terminates quickly and the final size is very small in the multiplex networks. Then, Lemma 4 is given.

**Lemma 4** *The time to global cascade in the underlying simplex network is less than the one in multiplex networks  $\{L_1, L_2, \dots, L_n\}$ .*

*Proof* Global cascade means that nearly all nodes in the networks can be activated. Supposing that the average probability of activating each node in a simplex network is  $p$  and the one of activating every conjoint agent is  $p'$ . The average influence received by the node in the simplex network is  $\tau_v$ . The average influences received by conjoint agents in multiplex networks are  $\tau_v^{L_1}, \tau_v^{L_2}, \dots, \tau_v^{L_n}$ . For the agent with the same identification, we have  $\tau_v \geq \tau_v^{L_1}$ ,  $\tau_v \geq \tau_v^{L_2} \dots \tau_v \geq \tau_v^{L_n}$  and  $\tau_v = \tau_v^{L_1} + \tau_v^{L_2} + \dots + \tau_v^{L_n} - \alpha$  where  $\alpha$  is the factor of influence redundancy. The probability of activating grows in direct proportion to the influences

received by each agent. Then,  $p = P(\tau_v \geq \theta_v) > p' = P(\tau_v^{L_1} \geq \theta_v || \tau_v^{L_2} \geq \theta_v \dots || \tau_v^{L_n} \geq \theta_v)$ . We use the  $\delta$  and  $\delta'$  to represent the final cascade sizes in the underlying simplex network and multiplex networks, respectively. According to the initial condition,  $\delta = \delta'$  and  $\delta/\delta' = 1$ . Then,  $\delta/\delta' \leq p/p'$ . After the transformation,  $\delta/p \leq \delta'/p'$ . Because  $p$  and  $p'$  are the average probability of activating each node, the times to reach global cascades in the underlying simplex network and multiplex networks can be represented as  $\delta/p$  and  $\delta'/p'$ , respectively.  $\square$

The above lemmas and corollary give upper bounds on the final size and velocity of the cascade process in multiplex networks and hold only if the parameter values of the agents' influences and threshold distributions in multiplex networks and the underlying simplex network are the same.

## 5 Simulation

The cascade process in multiplex social networks has been simulated on a computer. According to many previous studies [18,21,22,33,50], multiplex networks are constructed based on the Erdős–Rényi model [35], small-world model [36] and scale-free model [56]. Thresholds of nodes follow normal and uniform distributions.

The velocity and cascade size are two main parameters associated with the cascade process in networks. The velocity of the cascade process in networks is evaluated by comparing the time to reach the stationary state. The cascade size is measured by the average fraction of active nodes in a stationary state. Each trial is performed 500 times. The network is created in each of the 500 simulations, and the thresholds of agents are reassigned each time. All of the phenomena analyzed in the four case studies are simulated, and the theoretical results are validated.

To clarify the relationship between the velocity of information cascade and the network structure, some properties of the simplex network needs to be extended into the multiplex networks, such as the network density, average clustering coefficient, average path length and degree distribution [36,56]. The density  $D$  of the multiplex networks is defined as the ratio of the number of all the edges in multiplex networks to the number of the possible edges in multiplex networks, and is given by

$$D = \frac{\sum_{L_n} \sum_i |\Omega_{v_i}^{L_n}|}{(\sum_{L_n} |L_n|) [(\sum_{L_n} |L_n|) - 1]}.$$

The local clustering coefficient quantifies how close the neighbors of a node are to being a clique [36] and indicates the intensity of the reinforcements from neighbors in the cascade process [5]. In this paper, the local clustering coefficient  $C_{local}^{v'_i}$  of the conjoint agents  $v'_i$  is defined as the proportion of the number of the links between neighbors in all the layer of the multiplex networks divided by the number of the links that could possibly exist in the multiplex network [57]. Let  $C_{clique}(\Omega_{v_i}^{L_n})$  be the number of links between the neighbors of  $v_i^{L_n}$  in  $L_n$ . Then,

$$C_{local}^{v'_i} = \frac{2 \sum_{L_n} C_{clique}(\Omega_{v_i}^{L_n})}{\sum_{L_n} |\Omega_{v_i}^{L_n}| (|\Omega_{v_i}^{L_n}| - 1)}.$$

The average clustering coefficient  $C$  of the multiplex networks is the average of the local clustering coefficient of all the conjoint agents, and  $C = \sum_i C_{local}^{v_i} / |V|$ .

The average path length  $P$  is the measure of the efficiency of information cascade in networks and is defined by the average of the shortest paths between all pairs of agents in the multiplex networks [11]. The shortest paths in multiplex networks contain the cross-layers paths. Moreover, we use the graphs to represent the degree distribution of the multiplex networks. The degrees of all the agents in the multiplex networks are counted. The factor  $k$  represents the degree of the agent and  $p(k)$  represents the fraction of agents with degree  $k$  in networks [56].

### 5.1 Which is faster, simplex or multiplex?

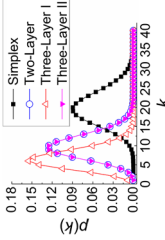
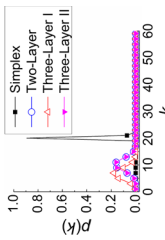
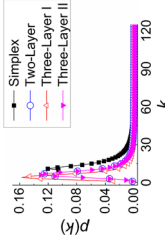
The main object of this section is to compare the cascade processes in simplex and multiplex networks. Layer structures and threshold distributions are varied in different trials. We assume that the influence of a node in a simplex network equals one unit. The vertical transfer coefficients are constant to ensure that the influences of a node in multiple layers equal the one in a simplex network.

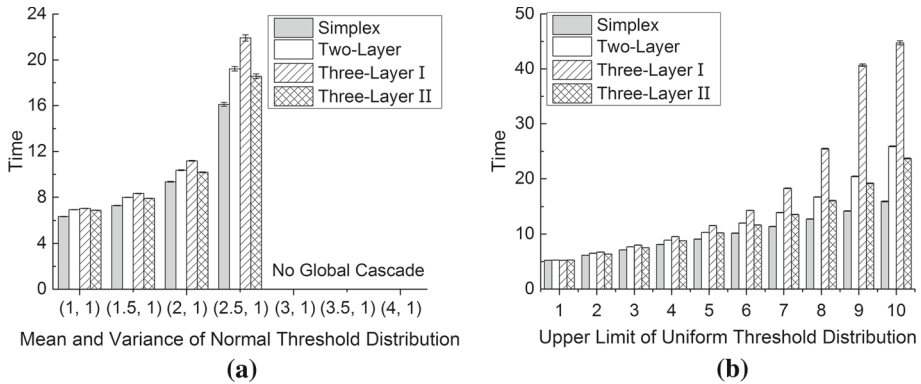
Based on the Erdős–Rényi model, small-world model and scale-free model, we first construct simplex networks with 10,000 nodes and an average of 20 neighbors. Meanwhile, in small-world simplex networks, the probability of interpolating between regular lattices is 0.1. In scale-free networks, the number of seed nodes is 50 with an average of 10 edges, and each subsequent node is added with 10 edges. Then, multiplex networks are generated: edges in a simplex network are distributed to multiple layers according to given probabilities. There are three multiplex networks. Two-layer and three-layer (type I) multiplex networks indicate the multiplex networks in the first case study: each layer contains 1/2 and 1/3 of edges in a simplex network, and no edges are allocated twice. Three-layer (type II) multiplex networks represent the network model in the second case study: each layer contains half of the edges in a simplex network, but the edges can be allocated repeatedly. After that, one node is randomly initialized as an active state. Then, information cascades are triggered in those simplex and multiplex networks independently. The lower limit of a uniform distribution is zero. The network properties of the simplex networks and the multiplex networks are summarized in Table 1.

From Figs. 7, 8, and 9, it can be clearly found that the cascade processes are slowed down in multiplex networks because the shortest paths in a simplex network are distributed in different layers. As shown in Table 1, the average path lengths in simplex networks are the lowest. When the threshold of a node is low, the times to reach stationary states in different networks are nearly the same because a node can be easily activated in any layer and a cross-layers cascade rarely takes place. However, as thresholds of nodes increase, the slow-down phenomenon in multiplex networks becomes more obvious. In this case, cross-layers cascade is more important because nodes with high thresholds become active only when enough neighbors are activated in other layers.

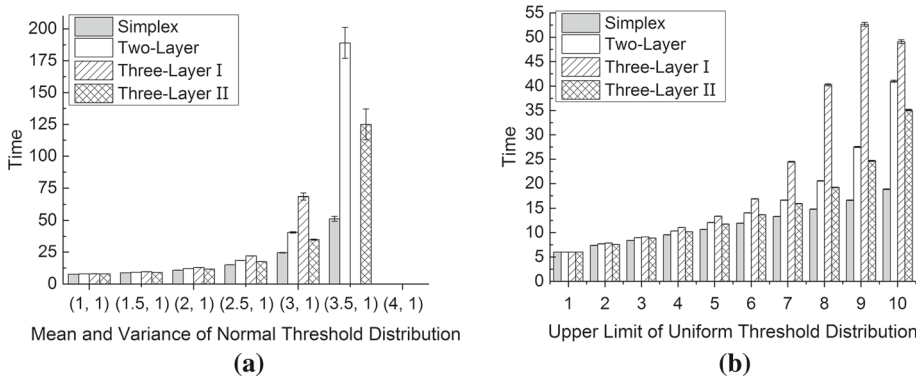
Meanwhile, information cascade in three-layer (type I) multiplex networks is slower than in two-layer multiplex networks. As shown in Table 1, the network density and the average clustering coefficient of three-layer (type I) multiplex networks are the lowest compared with simplex networks and the other two types of multiplex networks. Meanwhile, the average path lengths in three-layer (type I) multiplex networks are the longest. Therefore, the more layers are split from a simplex network, the slower information spreading in multiplex networks will be if each edge is allocated only once. With our cascade model, if the degree of a node

**Table 1** Properties of simulated networks

Types of networks	Network density	Average clustering coefficient	Average path length	Degree distribution
Simplex (Erdős–Rényi model)	0.00200	0.00238	3.408	
Two-layer (Erdős–Rényi model)	$5.00 \times 10^{-4}$	0.00138	4.141	
Three-layer I (Erdős–Rényi model)	$2.223 \times 10^{-4}$	0.00103	4.655	
Three-layer II (Erdős–Rényi model)	$3.335 \times 10^{-4}$	0.00138	4.119	
Simplex (small-world model)	0.00201	0.688	3.947	
Two-layer (small-world model)	$5.058 \times 10^{-4}$	0.344	4.877	
Three-layer I (small-world model)	$2.248 \times 10^{-4}$	0.229	5.460	
Three-layer II (small-world model)	$3.372 \times 10^{-4}$	0.344	4.858	
Simplex (scale-free model)	0.00209	0.00940	3.059	
Two-layer (scale-free model)	$5.218 \times 10^{-4}$	0.00472	3.688	
Three-layer I (scale-free model)	$2.319 \times 10^{-4}$	0.00315	4.109	
Three-layer II (scale-free model)	$3.478 \times 10^{-4}$	0.00474	3.684	



**Fig. 7** Time to reach stationary state with Erdős–Rényi model. **a** Mean and variance of normal threshold distribution. **b** Upper limit of uniform threshold distribution

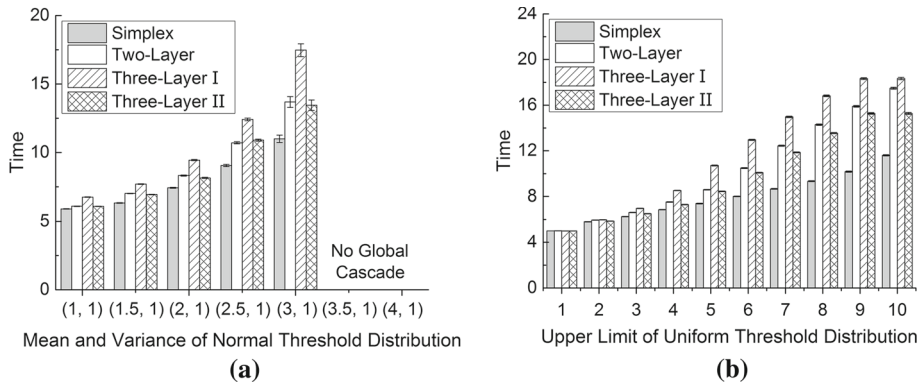


**Fig. 8** Time to reach stationary state with small-world model. **a** Mean and variance of normal threshold distribution. **b** Upper limit of uniform threshold distribution

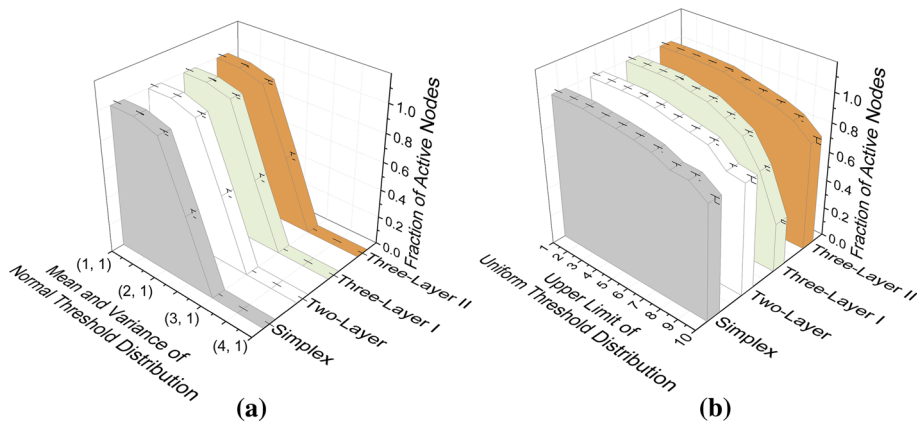
is large, a node can receive enough influences from neighbors to be activated more easily. On the contrary, if the average number of neighbors in one layer is comparatively small, activations rely more heavily on the cascades from other layers.

Compared with two-layer multiplex networks, one additional layer with the same average number of neighbors in three-layer (type II) multiplex networks provides extra paths of information propagation. As shown in Table 1, the additional layers do not vary the network densities, the average clustering coefficients and the degree distributions compared with two-layer multiplex networks. However, the average path lengths decrease but are larger than the one in simplex networks. Thus, cascade processes in three-layer (type II) multiplex networks are faster than in two-layer multiplex networks but are still slower than in simplex networks.

From Figs. 10, 11, and 12, multiplexity also restricts the final size of the cascade process. The inhibition effect of multiplexity becomes more obvious when thresholds of nodes become larger. Meanwhile, if network models and threshold distributions are different, multiplexity shows diverse inhibition effects on cascade processes. The fraction of active nodes largely decreases in three-layer (type I) multiplex networks with a uniform threshold distribution. However, with normal threshold distributions, the cascade size in the small-world network is larger than in the Erdős–Rényi and the scale free networks. The probable reason is that a



**Fig. 9** Time to reach stationary state with scale-free model. **a** Mean and variance of normal threshold distribution. **b** Upper limit of uniform threshold distribution



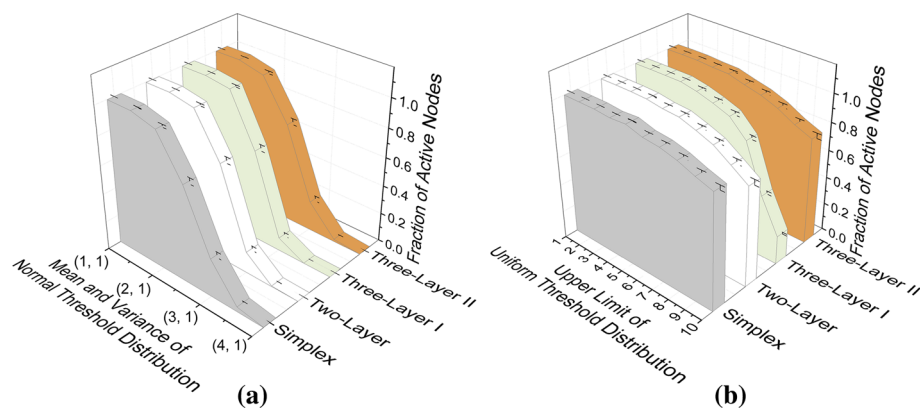
**Fig. 10** Average cascade size with Erdős–Rényi model

small-world network can provide more reinforcements from neighbors because of the high local clustering coefficient [5].

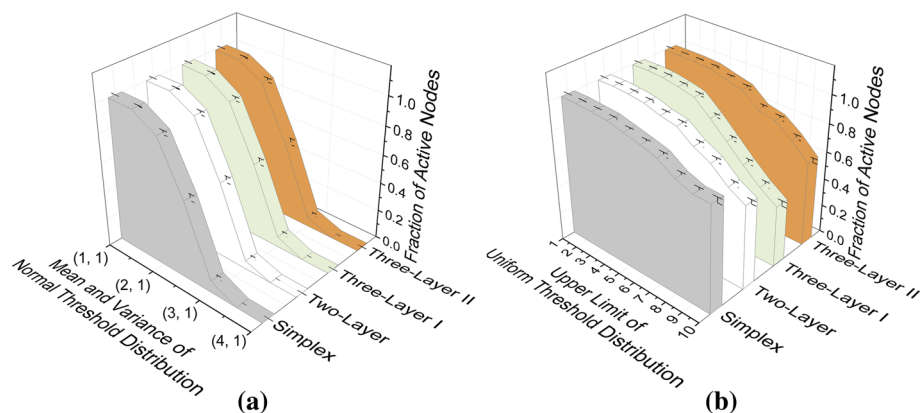
Therefore, the average path length of networks is closely related to the velocity of the cascade process. With different distributions of thresholds, the average clustering coefficient may relate to the size of the cascade process. For instance, the average clustering coefficients of the two-layer and the three-layer type II multiplex networks are similar and cannot indicate the variability of sizes of different cascade processes. However, the density and the degree distribution cannot be directly associated with the two features of the cascade process.

## 5.2 Effect of vertical transfer coefficient and multiplex structure

The main object of this section is to show the effect of the vertical transfer coefficient on the cascade process as analyzed in the third and fourth case studies. Meanwhile, the effect of the multiplex structure consisting of different network models is further investigated. There are two layers in the simulated multiplex networks. In the first series of trials, vertical transfer coefficients are positive, and two layers are generated by different network models.



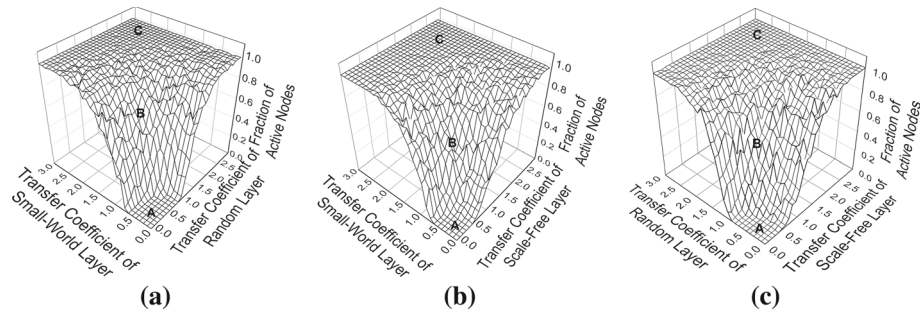
**Fig. 11** Average cascade size with small-world model



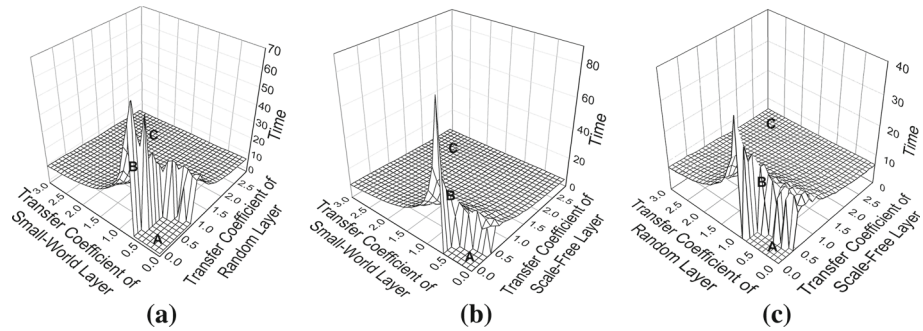
**Fig. 12** Average cascade size with scale-free model

$\lambda_{L_2}^{L_1}$  and  $\lambda_{L_1}^{L_2}$  are both varied from 0.1 to 3.0. The average number of neighbors in each layer is 10. In a small-world layer, the probability of interpolating between regular lattices is 0.1. In the scale-free layer, the number of seed nodes is 50 with an average of five edges, and each subsequent node is added with five edges. Each layer contains 10,000 nodes, and  $|V| = 10,000$ . Thresholds of nodes follow  $N(2.5, 1)$  normal distribution and  $U(0, 10)$  uniform distribution. The corresponding results are shown from Figs. 13, 14, 15, and 16.

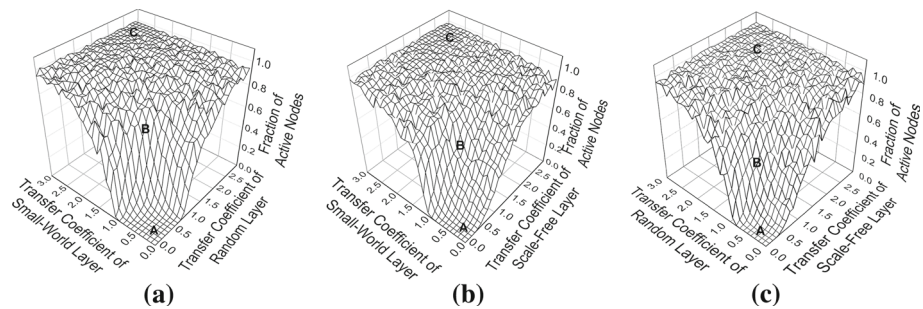
The surfaces in these figures are divided into three areas, according to the areas of parameter spaces in which different final sizes of cascade processes are found. A-area means that nodes are seldom activated and propagation dies out quickly. The low transfer coefficients of two layers restrict the cascade processes. The shape of A-area in Fig. 13 shows that the critical vertical transfer coefficient of a random layer between A-area and B-area is larger than the critical values of a small-world layer and scale-free layer if the vertical transfer coefficient of the conjoint layer is set as zero. Compared with the Erdős–Rényi network, the clustering coefficient of the small-world network is considerably larger, and one agent in a small-world network can receive more reinforcements if one active neighbor activates other agents [5]. It means that the Erdős–Rényi network with normal threshold distributions is not suitable for information spreading. However, the shape of the A-area in Fig. 15 indicates that



**Fig. 13** Average cascade size with normal threshold distribution:  $N(2.5, 1)$



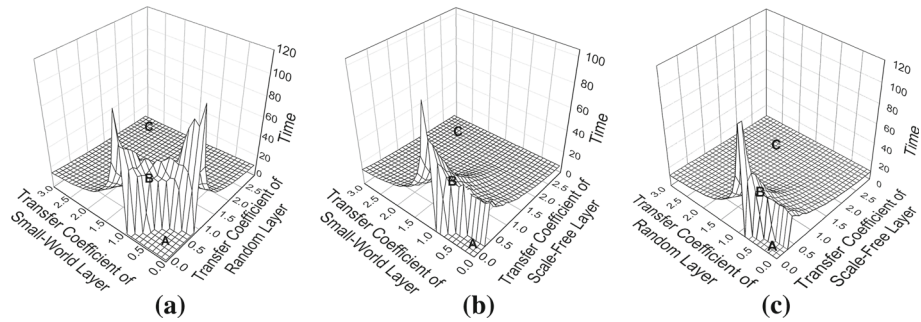
**Fig. 14** Time to reach stationary state with normal threshold distribution:  $N(2.5, 1)$



**Fig. 15** Average cascade size with uniform threshold distribution:  $U(0, 10)$

cascade processes in a random network and small-world network are similar but take place more quickly in a scale-free network with uniform threshold distributions.

The critical phenomena in Figs. 13 and 15 indicate that the scale free network is more vulnerable to the global cascade. However, reference [2] reports that the increased heterogeneity of degree distribution makes networks less vulnerable by comparing the cascade processes in scale free networks and Erdős–Rényi networks. The difference between our threshold model and the classical threshold model proposed in [2] is the underlying reason. In the classical threshold model, agents become active if enough fractions of the neighbors are activated. Therefore, hub nodes are hard to be activated because of the considerably larger amount of neighbors. Meanwhile, most nodes with very small amounts of neighbors are easily activated



**Fig. 16** Time to reach stationary state with uniform threshold distribution:  $U(0, 10)$

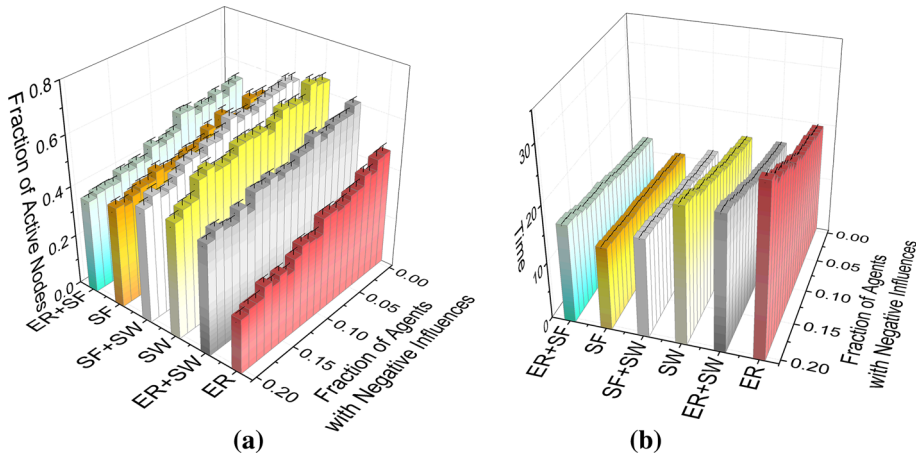
but have difficulty propagating the influence due to poor connections. In our threshold model, an agent is activated if it receives enough influences from active neighbors. Therefore, initial nodes with very small amounts of neighbors can still activate some hub nodes, and global cascades can emerge. Generally, hub nodes [10] are essential to the susceptibility of scale free networks with different types of threshold models.

As the transfer coefficients of two layers increase, more nodes gradually become active, and the times to reach final prevalence achieve peaks rapidly and then decrease sharply. This parameter space is named the B-area, which favors a fall in the graphs for a fraction of active nodes and a ridge in the graphs for the time to reach a stationary state. Many cascade processes in reality belong to the B-area, which is the transition region between global cascade and local popularity. In the B-area, multiplex networks contain many small groups of nodes, which are susceptible to the cascade process because of low thresholds but are separated by nodes with large thresholds. These separated groups of susceptible nodes mean that the sizes of the final prevalence may be different if different nodes are initialized as active to trigger the cascade processes in the same multiplex networks.

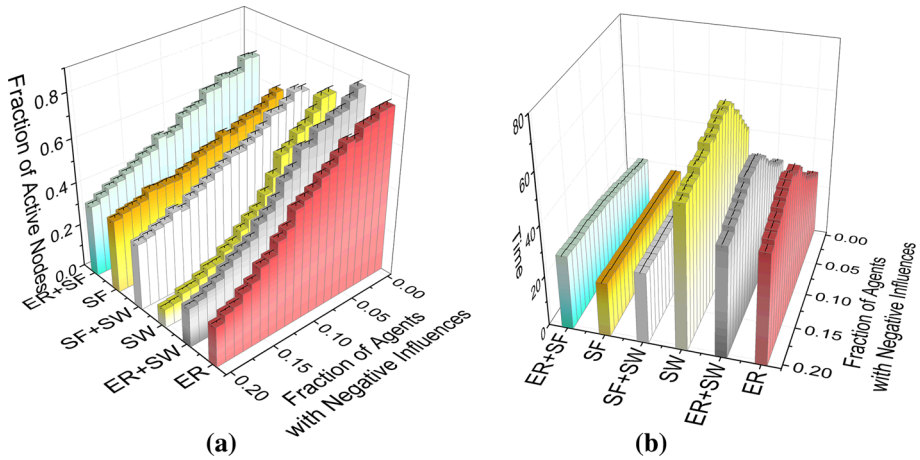
C-area means that the global cascades emerge smoothly. As the third case study analyzed, the large value of the vertical transfer coefficient leads to a rapid cascade in one layer. Because of the cross-layers cascades, a global cascade can take place in the multiplex networks even if the transfer coefficients of other layers are very small. Taking the online social network and word of mouth communication network as an instance, new fashions become widely known mostly because of the rapid spreading in an online social network.

In the second series of trials, vertical transfer coefficients for a part of agents in the duplex layers are set as negative. The fraction of agents with negative influences is varied from 0 to 0.2. The influences of other agents are set as one unit. The “splitting” and “combining” methods are both used to generate simulated multiplex networks. There are two layers in multiplex networks, and each layer has 10,000 nodes and an average of 10 neighbors. According to the model of simplex networks, multiplex networks generated by the “splitting” method are denoted by Erdős–Rényi (ER), small-world (SW) and scale-free (SF), respectively. Meanwhile, multiplex networks generated by the “combining” method are denoted by “ER + SW”, “SF + SW” and “ER + SF” according to the network models of duplex layers. Thresholds of nodes follow  $N(2.5, 1)$  normal distribution and  $U(0, 10)$  uniform distribution. The corresponding results are shown in Figs. 17 and 18. The properties of simulated networks are listed in Table 2. To save space, the graphs of degree distributions are not shown.

From Figs. 17 and 18, it can be generally found that the sizes of cascade processes decrease and the times to reach stationary states generally increase as the fraction of agents



**Fig. 17** Normal threshold distribution:  $N(2.5, 1)$



**Fig. 18** Uniform threshold distribution:  $U(0, 10)$

**Table 2** Selected properties of simulated networks

Types of networks	Network density	Average clustering coefficient	Average path length
ER + SF	$5.236 \times 10^{-4}$	0.00315	3.824
SF	$5.218 \times 10^{-4}$	0.00472	3.688
SF + SW	$5.252 \times 10^{-4}$	0.378	4.000
SW	$5.058 \times 10^{-4}$	0.344	4.877
ER + SW	$5.013 \times 10^{-4}$	0.340	4.403
ER	$5.00 \times 10^{-4}$	0.00138	4.141

with negative influences increases. The reduction of time to reach a stationary state in Fig. 18b means the cascade process quickly dies out instead of the high cascade velocity according to the cascade size shown in Fig. 18a. Meanwhile, the sizes of cascade processes decrease more rapidly if the thresholds of agents follow uniform distribution. If one layer in multiplex networks is a scale-free network, the times to reach stationary states are shorter and increase more slowly as the fraction of agents with negative influences increases. The probable reason is the robustness of scale-free networks with random failures [37] and shorter average path length compared with the SW, ER + SW and ER multiplex networks.

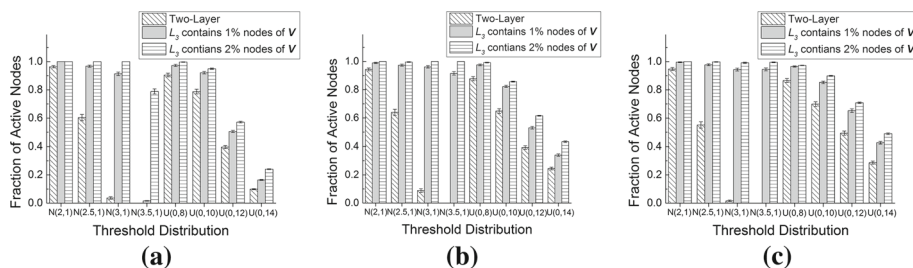
The ER + SW, ER + SF and SF + SW multiplex networks also have the main advantages of the small-world model and Erdős–Rényi model. Compared with ER multiplex networks with a normal threshold distribution, the sizes of cascade processes in ER + SW multiplex networks are considerably larger, and the times to reach stationary states are shorter. It is because that the small-world layer increases the average clustering coefficient of the multiplex networks as shown in Table 2. The cascade processes in ER + SW multiplex networks are similar to the ones in the ER multiplex networks if thresholds follow a uniform distribution. Meanwhile, the sizes of cascade processes in SF multiplex networks are less than the ones in SF + SW multiplex networks with a normal threshold distribution and in ER + SF multiplex networks with a uniform threshold distribution. Therefore, multiplex networks consisting of different network models are more competent for cascade process compared with multiplex networks generated by a single network model.

However, it can be found that the average clustering coefficient and average path length cannot be perfectly associated with the velocity and size of the cascade processes. For instance, the average path length of ER + SW multiplex networks is longer than the one of ER multiplex networks, but the cascade process take places more quickly in ER + SW multiplex networks with normal threshold distribution. Compared ER + SF multiplex networks and ER multiplex networks, the properties of networks can fit the two features of the cascade process. Oppositely, the clustering coefficient of ER + SF multiplex networks is less than the one of SF multiplex networks but the cascade sizes are larger. It means that the effects of the multiplex structure and the distribution of thresholds on the velocity and size of the cascade process are very complex. It requires more attention to propose more robust properties of the multiplex structure [31].

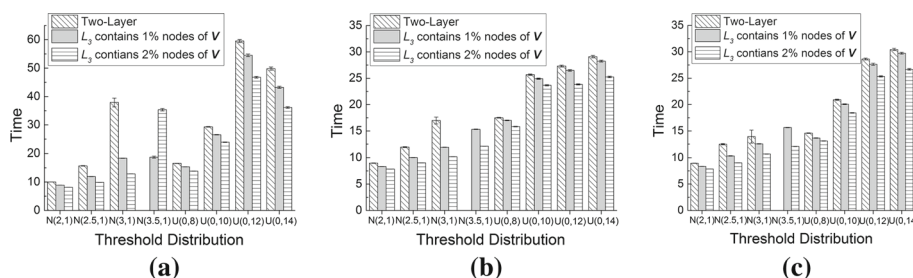
### 5.3 Leverage of tiny-scale layer

In this section, our aim is to present and discuss the leverage of a tiny-scale layer on the global cascade in multiplex networks: largely increasing the fraction of active nodes in a stationary state and reducing the time of the cascade process.

Multiplex networks contain two or three layers, and  $|V| = 10,000$ . In  $L_1$  and  $L_2$ , the average numbers of neighbors are both 10. According to the network models of  $L_1$  and  $L_2$ , simulated networks are named as ER + SW, ER + SF and SF + SW multiplex networks, respectively. In a small-world layer, the probability of interpolating between regular lattices is 0.1. In a scale-free layer, the number of seed nodes is 50 with an average of five edges, and each subsequent node is added with five edges. Each layer contains all nodes of  $V$ . One percent or two percent of nodes are randomly selected from  $V$  and constitute  $L_3$  according to the Erdős–Rényi model with an average three or six neighbors. Thus, the scale of  $L_3$  is very small, and connectivity is also sparse. Vertical transfer coefficients are constant. Influences of nodes in  $L_1$  and  $L_2$  are set as one unit. The vertical transfer coefficient of a tiny-scale  $L_3$  ( $\lambda^{L_3}$ ) has five units because we want to analyze the effects of quick propagation in a tiny-scale layer on other layers.  $L_3$  is only set as a random layer because propagations with



**Fig. 19** Leverage of tiny-scale layer on final size of cascade process: **a** ER + SW **b** SF + SW **c** ER + SF



**Fig. 20** Leverage of tiny-scale layer on time to reach stationary state of cascade process: **a** ER + SW **b** SF + SW **c** ER + SF

the three network models are similar (quick global cascading) when transfer coefficients are large as shown in Sect. 5.2. Threshold distributions are varied. One node in  $L_3$  is randomly set as the active state to trigger cascade processes in multiplex networks. The corresponding results are shown in Figs. 19 and 20. The selected properties of simulated networks are shown in Table 3.

As mentioned above, information spreads quickly in  $L_3$  because the influence of a node is large in  $L_3$ , while cascade processes in  $L_1$  and  $L_2$  are considerably slower. After adding  $L_3$ , the densities of multiplex networks slightly decrease. The average clustering coefficients and average path lengths generally remain unchanged. However, it can be found that the final sizes of cascade processes are raised from local popularity to global cascade by adding  $L_3$  into multiplex networks if thresholds of nodes follow normal  $N(3, 1)$  and  $N(3.5, 1)$  distributions. As analyzed in the second case study, more layers added into multiplex networks can facilitate the cascade process more or less if additional layers can provide extra short paths for information cascade. However,  $L_3$  is sparse and the number of nodes activated in  $L_3$  makes up a very small proportion of the final cascade size in multiplex networks. Global cascade takes place because separated but susceptible groups in  $L_1$  and  $L_2$  can be activated and conjoint. When nodes in  $L_3$  are activated, relational mapping nodes in  $L_1$  and  $L_2$  are activated because of cross-layers cascades. Then, nodes in susceptible groups are activated, and isolation caused by high threshold nodes gradually disappears. This facilitation effect is called the leverage of a tiny-scale layer on cascade processes in multiplex networks. Meanwhile, it can also be found that the facilitation effect is more obvious if one layer is a scale-free network, and thresholds follow a normal distribution. As shown in Fig. 19, cascade processes die out quickly with a  $N(3.5, 1)$  threshold distribution in all two-layer multiplex networks. The quick cascade process in  $L_3$  containing 1 % nodes of  $V$  can induce global

**Table 3** Selected properties of simulated networks

Types of networks	Network density	Average clustering coefficient	Average path length
ER + SW multiplex networks			
Two-layer	$5.013 \times 10^{-4}$	0.340	4.403
$L_3$ contains 1 % nodes of $V$	$4.972 \times 10^{-4}$	0.340	4.403
$L_3$ contains 2 % nodes of $V$	$4.945 \times 10^{-4}$	0.340	4.402
SF + SW multiplex networks			
Two-layer	$5.252 \times 10^{-4}$	0.378	4.000
$L_3$ contains 1 % nodes of $V$	$5.207 \times 10^{-4}$	0.377	4.000
$L_3$ contains 2 % nodes of $V$	$5.179 \times 10^{-4}$	0.376	4.006
ER + SF multiplex networks			
Two-layer	$5.236 \times 10^{-4}$	0.00315	3.824
$L_3$ contains 1 % nodes of $V$	$5.194 \times 10^{-4}$	0.00318	3.826
$L_3$ contains 2 % nodes of $V$	$5.165 \times 10^{-4}$	0.00329	3.820

cascade processes in SF + SW and ER + SF multiplex networks but only leads to a very small size of the cascade process in ER + SW multiplex networks.

However, if thresholds of nodes follow uniform distributions, the facilitation effect of a rapid cascade in  $L_3$  is limited. When the upper limits of uniform threshold distributions are larger than average degrees of  $L_1$  and  $L_2$ , many nodes cannot be activated even if all linking neighbors are active. Meanwhile, most nodes can be easily activated in two-layer multiplex networks if the upper limits of uniform threshold distributions are smaller than average degrees of  $L_1$  and  $L_2$ . In spite of this, the percentage increment of the fraction of active nodes by adding  $L_3$  is considerably larger than the scale of  $L_3$ .

The leverage of a tiny-scale layer is a complement to the previous studies on the facilitation of multiplexity. In [21], Brummitt et al. suggest that the cascade process in multiplex networks can be controlled by adding or removing the sparse layer, but the sparse layer contains the most parts of nodes in the dense layer and simulated multiplex networks. Our work suggests that a tiny-scale layer, which contains only one hundredth of nodes in multiplex networks, is also important. The superdiffusive behavior discussed in [20] means that the cascade process in multiplex networks is faster than in any disjoining layers. In our work, a fraction of active nodes in  $L_3$  reaches the stationary state much faster than in the multiplex networks. The scale of  $L_3$  is very small, and information propagates quickly. Cascade processes in some susceptible groups in  $L_1$  and  $L_2$  will not take place until the information has been transferred from  $L_3$ . In real social life, human behaviors always fall behind the cascade processes in online social networks. In spite of the time interval of cross-layers cascade, only prevailing

information, which has activated a large fraction of nodes in online social networks, can become the hot topics in a word of mouth network or even induce other collective behaviors, such as panic buying and a protest movement. The leverage of the tiny-scale layer is also different from the effect of hub nodes in networks [10]. Hub nodes have a considerably larger amount of neighbors than other nodes in networks. Nodes in  $L_3$  are randomly selected and have a similar number of neighbors in the underlying simplex network. Moreover, if the influences of agents vary across layers, the properties of network structure can be hardly used to evaluate the cascade process in multiplex networks.

## 6 Discussion and future work

In this paper, we focus on the role of a cascade across layers in the information propagation in multiplex networks. The mapping relationship and vertical transfer coefficient are proposed to be the main features of the cross-layers cascade: one conjoins multiple layers and provides the paths for information spreading between layers; and the other one quantifies the influences of one node varied in multiple layers. After providing the generalized linear threshold cascade model, we analyzed how multiplexity slows down or speeds up information cascades based on the cross-layers cascade.

The main reason for the slow-down phenomenon of information spreading is the obstruction of cross-layers cascade, which connects the distributed shortest path in multiple layers. When the information spreading on the first part of the shortest path in one layer cannot be transferred to the next intermediate nodes in other layers, downstream nodes can only be activated by the cascade processes on other topologically inefficient paths. As a result, the time to reach global cascade in multiplex networks is longer than in simplex networks. However, the topologically inefficient path reported in the research of empirical data [22] may be the most efficient in the framework of multiplex networks because information selectively propagates on networks. On the other hand, information can spread in a particular part of social agents more pertinently with the consideration of diverse linking types (multi-relation). For example, Google+ allows users to arrange neighbor nodes and share information in different “Circles” according to different relationships. “Circles” restrict the velocity and range of spreading information but help to protect the privacy of users and avoid the troubles caused by a wide dissemination of information with no restriction. In other words, cross-layers cascade can be carefully controlled by users with the aid of these subjectively created “Circles”.

The inhibitory effect of a negative agent’s influence is the other reason for the slow-down phenomenon. To activate linking neighbors, more positive influences are needed to cancel out the negative influences with the generalized linear threshold model. Therefore, the time to reach the stationary state is delayed, or the final size of the cascade process decreases. The negative influence generally represents the adverse opinion on a product [54], and the adverse opinion cannot be disseminated in this paper. However, multiplex networks provide a new framework to analyze the issue of competitive cascade processes, which are more closely coupled because of the cross-layers paths. By extensive simulations, it is found that the inhibitory effect of negative influence can be mitigated if one of the multiple layers is a scale-free network. The robustness of scale-free networks with random failures [37] is the probable reason. It can also be found that the cascade size is larger with a uniform threshold distribution if one layer is the Erdős–Rényi network, and the information spreads more widely with a normal threshold distribution if one layer is the small-world network. Moreover, a scale-free layer in multiplex networks can make the facilitation effect of a tiny-

scale layer more obvious if thresholds follow a normal distribution. Therefore, multiplex networks consisting of different network models are more competent for the cascade process compared with multiplex networks generated by a single network model. It is suggested that combining suitable networks together may be a feasible approach to controlling the cascade process.

Extra short paths and rapid spreading in the additional layer can both facilitate cascade processes in multiplex networks compared with disjointed layers. The effect of popular online social networks on information spreading is similar to the one of additional layers conjoining to traditional communication networks because users can make friends and share information with strangers conveniently in online social networks. The leverage of a tiny-scale layer on global cascade indicates the difficulties of predicting or controlling the cascade process in multiplex networks. Because of the cascade across layers, nodes activated in a tiny-scale layer can trigger concurrent cascade processes in the conjoining large-scale layer if there are many susceptible but separated groups in multiplex networks.

The issue of information cascade in multiplex social networks may provide a basis for further exploration in other multi-agent systems, such as normative multi-agent systems [58], trust systems [59] and artificial agent societies [60], where the role of simplex network topologies has been widely investigated. Similar to the different speeds of information cascades in multiple layers, the time to reach convention [58] or the rule of norm evolution [60] in each layer of multiplex networks may also be different. Meanwhile, the trust path for the selection of trustworthy service analyzed in [59] is probably distributed in multiple layers and connected by many cross-layers paths. The effect of multiplex networks generated by a single or different network models on these multi-agent systems is still an open question. The timescales for formulating multiplex networks vary greatly according to different real-world systems. For instance, multiple relationships between human in social networks are inherent [15] but integrated cyber-physical systems emerge because of the progressive development of underpinning technology for major industries [9]. Generally, traditional systems can be represented as multiplex networks if the inherent multiple relationships greatly affect the interactions between agents [15] or several multi-agent systems are functionally interdependent [19].

In future work, we intend to provide a more detailed description of the cross-layers cascade and apply it to real multiplex networks. A formalized description of the cascade process in multiplex networks depending on the cross-layers cascade is needed. In addition, the effects of threshold distribution and layer structure on the cascade process will also be further analyzed. Moreover, the timescale of the cross-layers cascade and the communication between nodes in single layer may also lead to topologically inefficient paths if some nodes in the shortest paths are temporally incapable of the communication [32]. Research on multiplex networks is attracting more attention, but real field data of the cascade process in multiplex social networks are still rare. The main difficulties are how to judge social agents that are conjoint in different networks and track the information spreading on and across networks together. We anticipate that the concept of cross-layers cascade can inspire the additional study of information spreading in multiplex networks.

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